MOTION AND FILLING OF CAVITIES IN A BOUNDLESS LIQUID AND CLOSE TO A PLANE

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The motion of bubbles in liquids has been studied in many earlier papers [1-8]. In this paper methods of the projection type are applied to the problem of a cavity in an ideal, incompressible liquid in the absence of vortices. The collapse of a bubble having a finite initial velocity in a boundless liquid is considered; also considered is the collapse of a stationary bubble close to a solid wall. Using the small-parameter method the generation of a jet is examined analytically. A numerical computing method not involving small parameters is developed; it is based on calculating the projection by numerical computation of the corresponding integrals. The method combines economy and simplicity of application with a high accuracy in the region in which the representation of the velocity potential by a series of spherical functions remains effective.

1. Filling of a Cavity Moving in a Boundless Liquid. At the instant $t = 0$ we consider a spherical cavity of radius $l = 1$ moving at a velocity $u \ll |l|$, its motion being described by a dipole and sink in the center of the sphere. The pressure equals zero inside the cavity and unity at an infinite distance. We represent the contour of the cavity by a series in Legendre polynomials and the velocity potential Φ by a series in spherical functions:

$$R = l\left(1 + \sum_{n=1}^{\infty} d_n P_n(\cos\theta)\right);$$

$$\Phi = \sum_{n=0}^{\infty} \frac{c_n l^n}{r^{n+1}} P_n(\cos\theta). \tag{1.1}$$

Here r is the distance from the center, which moves at a velocity u(t); θ is the angle reckoned from the direction of motion. On the contour of the cavity $r = R$

$$\frac{\partial R}{\partial t} + u\left(\cos\theta + \frac{\partial R}{\partial\theta}\frac{\sin\theta}{R}\right) = \frac{\partial\Phi}{\partial r} - \frac{1}{R^2}\frac{\partial R}{\partial\theta}\frac{\partial\Phi}{\partial\theta}. \tag{1.2}$$

The Cauchy–Lagrange integral on the boundary in the moving system gives

$$\frac{\partial\Phi}{\partial t} + \frac{u}{R}\sin\theta\frac{\partial\Phi}{\partial\theta} - u\cos\theta\frac{\partial\Phi}{\partial r} + \frac{1}{2}|\nabla\Phi|^2 = 1. \tag{1.3}$$

There is an arbitrary choice of u for $t > 0$. Assuming that $|u| \ll |l|$, we may seek the solution by way of successive approximations. If to a first approximation $d_1 = 0$, we find that with respect to $\varepsilon = u/l$ the coefficients in Eq. (1.1) are of the following orders:

$$d_1 \sim \varepsilon^3,\ d_2 \sim \varepsilon^2,\ d_3 \sim \varepsilon^3;$$
$$c_1 \sim \varepsilon,\ c_2 \sim \varepsilon^2,\ c_3 \sim \varepsilon^3. \tag{1.4}$$

For $n \geq 4$ the coefficients $d_n = 0(\varepsilon^3)$. Neglecting small quantities of the form ε^4, from (1.1)-(1.2), we obtain

$$c_0 = -l^2 l^{\cdot};\ 2c_1 = ul^2(3/5d_2 - 1) - (l^3 d_1)^{\cdot};\ 3c_2 = -(l^3 d_2)^{\cdot};$$

$$c_3 = -1/4(l^3 d_3)^{\cdot} - 9/10 l^2 u d_2;\ c_n = -(l^3 d_n)^{\cdot}/(n+1). \tag{1.5}$$

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Here $n \geq 4$. We write the equation for the velocity u in the form

$$3ul^{\cdot} + lu^{\cdot} = Q, \quad Q = 0(\varepsilon^2). \tag{1.6}$$

Substitution of (1.5) and (1.6) into (1.3) after the projection of the resultant expression on the Legendre polynomials enables us – to an accuracy of small quantities of the order of $0(\varepsilon^4)$ – to derive equations for the rate of collapse and the amplitudes of deformation:

$$\begin{aligned}
&3/2l^{\cdot 2} + ll^{\cdot\cdot} = 1/4u^2 - 1;\\
&A_1 = 3/5\left(3ld_2^{\cdot} - 4l^{\cdot}d_2\right)u - Q\,(1 - 7/5d_2);\quad A_2 = -9/4u^2;\\
&A_3 = -6\left(12/5l^{\cdot}d_2 + ld_2^{\cdot}\right)u - 6/5Qd_2;\\
&A_n \equiv l^2 d_n^{\cdot\cdot} + 5ll^{\cdot}d_n^{\cdot} + 3l^{\cdot 2}d_n + (2-n)\,ll^{\cdot\cdot}d_n.
\end{aligned} \tag{1.7}$$

The equations $A_n = 0$ (with $n \geq 4$) correspond to the equations governing the linear theory of stability of a collapsing bubble [3, 4].

We see from (1.7) that the deformations of the moving and collapsing cavity are excited in a cascade manner. As a result of the forward velocity, flattening occurs along the axis of motion (i.e., $d_2 < 0$), and by virtue of the simultaneous presence of flattening and forward (translational) velocity alone we have $d_3 \neq 0$.

The first equation of (1.7) and Eq. (1.6) correspond (for $Q = 0$) to the model problem regarding a moving sphere of variable volume [8]; in the collapse of this sphere an instant necessarily arises at which $|u| \sim |l|$. On approaching this instant, Eqs. (1.7) cease holding, since their solutions d_2 and d_3 are no longer small.

Asymptotic analysis of Eqs. (1.7) in the range $l \ll 1$, in which the solution $l(t)$ may be expressed in power form, shows that $d_3 > 0$. This corresponds to the formation of a jet in the "tail" region $\theta \sim \pi$.

2. Cavity Close to a Plane. Let a cavity stationary at $t = 0$ lie at a fair distance from the plane so that $l \ll a$, where a is the distance from the origin of coordinates to its image in the plane. We shall seek the velocity potential by an alternating method. In order to obtain a solution to an accuracy of ξ^5 it is sufficient to have two approximations

$$\begin{aligned}
&\Phi = \Phi_0^1 + \Phi_0^2 + \Phi_1^1 + \Phi_1^2 + \ldots\\
&\Phi_0^i = \sum_{n=0}^{\infty} \frac{c_n l^n}{r_i^{n+1}} P_n(\cos\theta_i),\\
&\Phi_1^i = \sum_{n=0}^{\infty} \frac{c_n' l^n}{r_i^{n+1}} P_n(\cos\theta_i).
\end{aligned} \tag{2.1}$$

Here i is the index of the cavity or its image. The polar axes of the coordinate systems are directed toward one another. On the surface of the cavity i, $k = 1, 2$

$$\left(\frac{\partial}{\partial r_i} - \frac{1}{r_i^2}\frac{\partial R_i}{\partial\theta}\frac{\partial}{\partial\theta_i}\right)\left(\Phi_1^i + \Phi_0^k\right) = 0; \quad i \neq k. \tag{2.2}$$

From Eqs. (2.1), (2.2), and (1.5) we may derive

$$\begin{aligned}
&c_0' = 0;\quad c_1' = -1/2\xi^2 l^2\,[l^{\cdot}(1 - 3/5d_2) + u\xi];\\
&c_2' = -2/3l^{\cdot}l^2\xi^3;\quad c_3' = -l^{\cdot}l^2\xi^2\,(3/4\xi^2 + 9/10d_2).
\end{aligned} \tag{2.3}$$

Here and subsequently we omit terms of the order $\varepsilon^n\xi^m (m > 0)$, if $n + m > 4$ or of the order ε^n if $n > 3$. The coefficients $c_n' = 0$ for $n \geq 4$. In order to make $d_1 = 0(\xi^2)$, we must choose Q in (1.6) in the following way:

$$3ul^{\cdot} + lu^{\cdot} = -3\xi^2(3l^{\cdot 2} + l^{\cdot\cdot}l) = Q. \tag{2.4}$$

Substituting Eqs. (2.1) with coefficients (2.3) into (1.3) and allowing for (2.4) and Eq. (1.7), which are valid for $\xi = 0$, we obtain

$$\begin{aligned}
&ll^{\cdot\cdot}(1 + \xi - \xi^4) + 2l^{\cdot 2}(3/4 + \xi + \xi^4) + 1 = 1/4u^2 - 1/2ul^{\cdot}\xi^2;\\
&5/3A_1 = \left(3ld_2^{\cdot} - 4l^{\cdot}d_2\right)u + \xi^2\,[3ll^{\cdot}d_2 - 6\,(2l^{\cdot 2} + ll^{\cdot\cdot})\,d_2 - 19ul^{\cdot}\xi];\\
&A_2 = -9/4u^2 - \xi^2\,(20\xi l^{\cdot 2} + 5\xi l^{\cdot\cdot}l - 9/2ul^{\cdot});\\
&1/6A_3 = -6/5ul^{\cdot}\,(2d_2 + \xi^3) - ld_2^{\cdot}\,(u + l^{\cdot}\xi^2) - 18/5d_2\xi^2 l^{\cdot}.
\end{aligned} \tag{2.5}$$

The coefficients $d_n = 0$ for $n \geq 4$, since they satisfy the homogeneous linear equations with zero initial conditions.

The first equation of (2.5) and Eq. (2.4) correspond to the equations describing the model of a sphere of variable volume [8]. Deformation starts influencing the collapse and motion of the bubble in approximations higher than (2.5). As a result of interaction with the plane, the stationary cavity first acquires a velocity (directed toward the plane), increasing in the course of collapse to reach values exceeding the rate of change of the radius [8]. A limiting radius of collapse may always be found. The solution to the last three equations of (2.5) may be written in approximate form for short times and also for small radii $l \ll 1$. It follows from the third equation of (2.5) that at the onset of the collapse $d_2 > 0$, i.e., the cavity is drawn out in the direction of the plane. The last equation of (2.5) gives $d_3 < 0$ at the onset of collapse, i.e., the cavity acquires an oval form. As the radius of the cavity diminishes, a moment arises at which $d_3 > 0$, and in the neighborhood of the boundary furthest removed from the plane the curvature diminishes, i.e., a jet is formed. From a particular instant the amplitudes of d_2 and d_3 become so large that Eqs. (2.5) lose their validity.

3. Cavity in a Boundless Liquid. Projection Method. Let us consider the construction of a numerical method for describing the motion of a cavity without having to introduce any small parameter. One of the shortcomings of existing methods of calculation based on spherical functions such as [6] is the highly cumbersome nature of the equations, which increases catastrophically with increasing number of terms in the basic expansions. We may derive a far simpler and more efficient method if we make no attempt to expand the cumbersome products of sums but calculate the projections directly.

The velocity potential Φ in the absolute system coinciding at a specified instant with the system moving at a velocity u may be expressed in the form

$$\Phi = \sum_{n=0}^{\infty} F_n(t) \frac{P_n(z)}{r^{n+1}}, \quad z = \cos\theta. \tag{3.1}$$

The coordinates of a point on the surface of the cavity are given by

$$R = \sum_{n=0}^{\infty} R_n(t) P_n(z). \tag{3.2}$$

If we introduce the velocity potential φ in a system moving with the velocity u

$$\varphi = \Phi - urz, \tag{3.3}$$

the kinematic and dynamic conditions ar $r = R$ take the form

$$\frac{\partial R}{\partial t} = \frac{\partial \varphi}{\partial r} - \frac{1 - z^2}{R^2} \frac{\partial \varphi}{\partial z} \frac{\partial R}{\partial z}; \tag{3.4}$$

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2}\left(\frac{\partial \varphi}{\partial r}\right)^2 + \frac{1 - z^2}{2R^2}\left(\frac{\partial \varphi}{\partial z}\right)^2 = 1 + \frac{u^2}{2}. \tag{3.5}$$

On substituting (3.1) and (3.2) into (3.4) and (3.5) and projecting onto the Legendre polynomials, we obtain the system of equations

$$\frac{dR_n}{dt} = \left(n + \frac{1}{2}\right) \int_{-1}^{1} \Omega P_n dz; \quad \sum_{k=1}^{\infty} G_{nk} \frac{dF_k}{dt} = \int_{-1}^{1} \Gamma P_n dz;$$

$$G_{nk} = \int_{-1}^{1} \frac{P_k P_n}{R^k} dz; \quad k, n = 0, 1, \ldots$$

$$\Omega = uz + \sum_{n=0}^{\infty} F_n \frac{n+1}{R^{n+2}} P_n + \frac{1 - z^2}{R}\left(u - \sum_{n=0}^{\infty} \frac{F_n}{R^{n+2}} P_n'\right) \sum_{n=0}^{\infty} R_n P_n';$$

$$2\Gamma = 2 + u^2 - \left(uz + \sum_{n=0}^{\infty} F_n \frac{n+1}{R^{n+2}} P_n\right)^2 + (1 - z^2)\left(u - \sum_{n=0}^{\infty} \frac{F_n}{R^{n+2}} P_n'\right)^2. \tag{3.6}$$

Each integral in Eqs. (3.6) is a single-valued function of the variables $R_0, R_1, \ldots, F_0, F_1, \ldots$, so that the system of equations (3.2) and (3.6) represents an infinite system of ordinary differential equations, linear with respect to the derivatives. The function u(t) entering into (3.6) via (3.3) may be chosen arbitrarily. It is only essential that the origin of coordinates should remain permanently inside the cavity, a reasonably long way from its surface. We may assume, in particular, that $u = u_0 l^{-3}$ or determine u from the condition $R_1 = 0$.

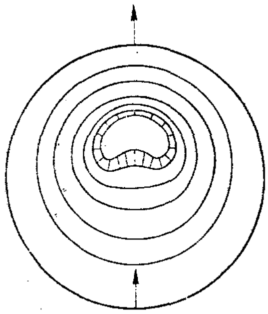

Fig. 1

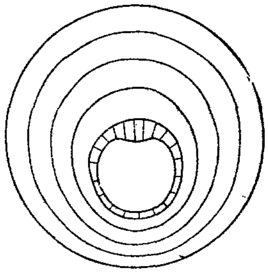

Fig. 2

At the initial instant the cavity had the shape of a sphere and was moving at a velocity u_0:

$$u=u_0;\ R_0=1,\ R_1=0,\ \ldots;\ F_0=0,\ F_1=-1/2,\ F_2=0,\ldots;\ t=0. \tag{3.7}$$

The system (3.6) with initial conditions (3.7) was solved by approximating it to a finite system of order m. It was assumed that $R_n=0$, $F_n=0$, for $n>m$. The system of ordinary differential equations of order 2m derived from (3.6) was reduced at each step in t to a system of equations resolved with respect to the derivatives; this was integrated by the Runge–Kutta method. Calculation of the integrals in (3.6) incorporating Legendre polynomials and their derivatives was carried out by Simpson's rule, the step being taken as ~1/8 of the distance between the nearest zeros of the leading polynomial $P_m(\cos\theta)$. For monitoring purposes, at each time step we calculated the total energy of the system, which was equal to the sum of the potential energy of the cavity, proportional to its volume, and the kinetic energy of the liquid

$$T=\pi\int_{-1}^{1}\Phi\left[R\left(\frac{\partial R}{\partial t}+uz\right)+u(1-z^2)\frac{\partial R}{\partial z}\right]Rdz.$$

Here Φ are the values of the potential on the surface.

The calculations were carried out in the MINSK-2 computer. The results of calculations with 16 Legendre polynomials and $u_0=0.1$ are presented in Fig. 1. The form of the cavity is illustrated at the instants of time t = 0; 0.64; 0.76; 0.84; 0.875; 0.895. For the instant t = 0.895, Fig. 1 also shows the distribution of normal velocity $\partial\Phi/\partial n$ along the boundary of the cavity. For t = 0.895 the velocity at the point $\theta=\pi$ on the surface is more than three times the velocity at the point $\theta=0$. The front part of the cavity ($\theta\sim 0$) deviates little from spherical in the course of collapse. In the rear section of the cavity at $t\approx 0.87$ the curvature passes through zero, and a broad jet is formed. The deformations are large, since the shape of the cavity in the region $\theta\sim\pi$ differs considerably from that in the region $\theta\sim 0$. For several "dents" on the surface of the cavity the velocity potential of the liquid cannot be expressed in the form (3.1). The error in the calculations accordingly increases when the curvature in the region of $\theta\sim\pi$ is negative. In the example presented the error in the energy integral reaches 4% at t = 0.895, although for $t\sim 0.8$ the error is less than $3\cdot 10^{-5}$. It should be noted that an error of 4% in the energy represents a reasonable accuracy, since the error in the coordinates of the cavity boundary is here much smaller than the error in the boundary velocity.

We also carried out some calculations for the case of a stationary origin of coordinates, u = 0; the results agreed with the case $u\neq 0$. The choice of velocity u had no effect on the character of the solution. However, over a certain range the error in the energy integral diminished. By choosing a fairly small time step in the integration we were able to make the energy error less than 10^{-6} in the range $t<0.865$.

4. Projection Method in Analyzing the Collapse of a Cavity Close to a Plane. The motion of the cavity in the liquid close to a plane is equivalent to the motion of two cavities lying symmetrically on either side of the plane in a boundless liquid. In the boundary conditions on the surface of the cavity we must allow for the contribution due to a potential with singularities in the neighboring plane (subsequently, we shall give the index 1 to quantities associated with the neighboring plane, while for the cavity under consideration the notation employed in Sec. 3 remains valid). The additional term in the velocity potential due to the presence of the neighboring plane equals

$$\Phi_1=\sum_{n=0}^{\infty}\frac{E_n(t)}{r_1^{n+1}}P_n(z_1). \tag{4.1}$$

Allowing for this term in the boundary conditions (3.4) and (3.5), we may derive equations similar to (3.6), simply differing in the values of Ω, Γ, G_{nk}

$$G_{nk} = \int_{-1}^{1} \left[\frac{P_k(z)}{R^k} + \frac{P_k(z_1)}{r_1^{\,k}} \right] P_n(z)\, dz;$$
$$\Omega = -W_1 - \frac{1-z^2}{R} W_2 \frac{\partial R}{\partial z};$$
$$\Gamma = 1 + \frac{1}{2}\left[u^2 - W_1^{\,2} - (1-z^2) W_2^{\,2}\right] + \frac{2u}{r_1}\left(1 - z_1^{\,2}\right)\frac{\partial \Phi_1}{\partial z_1} + 2uz_1 \frac{\partial \Phi_1}{\partial r_1};$$
$$W_1 = uz + \sum_{n=0}^{\infty} F_n \frac{n+1}{R^{n+2}} P_n(z) - \frac{R - az}{r_1}\frac{\partial \Phi_1}{\partial r_1} + \frac{1-z^2}{r_1^{\,3}} Ra \frac{\partial \Phi_1}{\partial z_1}; \tag{4.2}$$
$$W_2 = -u + \sum_{n=0}^{\infty} F_n \frac{P_n'(z)}{R^{n+2}} - \frac{a}{r_1}\frac{\partial \Phi_1}{\partial r_1} - \frac{R}{r_1^{\,3}}(R - az)\frac{\partial \Phi_1}{\partial z_1};$$
$$r_1^{\,2} = (a^2 - 2aRz + R^2)^{1/2};\quad z_1 = (a - Rz)/r_1;\quad da/dt = -2u.$$

The velocity u of the origin of coordinates may be determined from the conditions $R_1(t) = 0$, which gives $\int_{-1}^{1} \Omega P_1(z)\, dz = 0.$

Together with the right-hand sides and coefficients of (4.2), Eqs. (3.6) represent an infinite system of differential equations in R_0, R_1, ..., F_0, F_1, ..., linear with respect to the derivatives. The method of solution was described in Sec. 3. The results of our calculation of the shape of the cavity are presented in Fig. 2 for an initial distance of $a_0 = 3$ and an instant of time t = 0; 0.62; 0.82; 0.935; 0.99. For the instant t = 0.99, Fig. 2 shows the distribution of normal velocity along the boundary.

We see that as a result of its collapse the cavity is accelerated toward the plane. This effect was earlier discussed on the basis of the model based on a sphere of variable volume [8]. The velocity at the surface point $\theta = \pi$ is much greater than the velocity at the point $\theta = 0$, i.e., a jet is created, directed toward the plane. The formation of the jet may be seen from the denting of that part of the cavity surface further removed from the plane (Fig. 2).

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